Learning Temple
IIT/NEET ACADEMY

25th Feb. 2021 | Shift - 2
MATHEMATICS

SECTION-A

1. A plane passes through the points A(1, 2, 3), B(2, 3, 1) and C(2, 4, 2). If O is the origin and P is (2, −1, 1), then the projection of $\overrightarrow{OP}$ on this plane is of length:

(1) $\sqrt{\frac{2}{5}}$

(2) $\sqrt{\frac{2}{3}}$

(3) $\sqrt{\frac{2}{11}}$

(4) $\sqrt{\frac{2}{7}}$

Ans. **(3)**

Sol. A(1, 2, 3), B(2, 3, 1), C(2, 4, 2), O(0, 0, 0)

Equation of plane passing through A, B, C will be

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2-1 & 3-2 & 1-3 \\ 2-1 & 4-2 & 2-3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 0$$

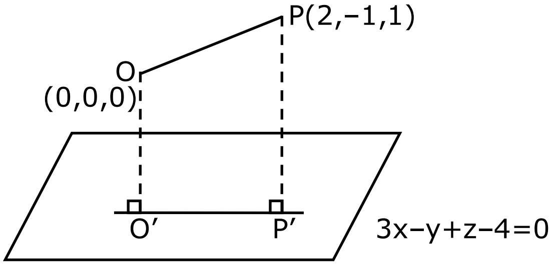
$\Rightarrow$ (x − 1)(−1 + 4) − (y − 2)(−1 + 2) + (z − 3)(2 − 1) = 0

$\Rightarrow$ (x − 1)(3) − (y − 2)(1) + (z − 3)(1) = 0

$\Rightarrow$ 3x − 3 − y + 2 + z − 3 = 0

$\Rightarrow$ 3x − y + z − 4 = 0, is the required plane.

Now, given O(0, 0, 0) & P(2, −1, 1)



Plane is 3x − y + z − 4 = 0

O′ & P′ are foot of perpendiculars.

for O'

$$\frac{x-0}{3}=\frac{y-0}{-1}=\frac{z-0}{1}=\frac{-(0-0+0-4)}{9+1+1}$$

$$\frac{x}{3}=\frac{y}{-1}=\frac{z}{1}=\frac{4}{11}$$

$$\Rightarrow \quad O'\left(\frac{12}{11},\frac{-4}{11},\frac{4}{11}\right)$$

for P'

$$\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-1}{1}=\frac{-(3(2)-(-1)+1-4)}{9+1+1}$$

$$\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-1}{1}=\left(\frac{-4}{11}\right)$$

$$P'\left(\frac{-12}{11}+2,\ \frac{4}{11}-1,\ \frac{-4}{11}+1\right)$$

$$\Rightarrow \quad P'\left(\frac{10}{11},\frac{-7}{11},\frac{7}{11}\right)$$

$$O'P'=\sqrt{\left(\frac{10}{11}-\frac{12}{11}\right)^2+\left(\frac{-7}{11}+\frac{4}{11}\right)^2+\left(\frac{7}{11}-\frac{4}{11}\right)^2}$$

$$\Rightarrow \quad O'P'=\frac{1}{11}\sqrt{4+9+9}$$

$$\Rightarrow \quad O'P'=\frac{\sqrt{22}}{11}$$

$$\Rightarrow \quad O'P'=\frac{\sqrt{2}\times\sqrt{11}}{11}$$

$$\Rightarrow \quad O'P'=\sqrt{\frac{2}{11}}$$

2. The contrapositive of the statement "If you will work, you will earn money" is:

(1) If you will not earn money, you will not work

(2) You will earn money, if you will not work

(3) If you will earn money, you will work

(4) To earn money, you need to work

Ans. (1)

Sol. Contrapositive of $p \to q$ is $\sim q \to \sim p$

$p \to$ you will work

$q \to$ you will earn money

$\sim q \to$ you will not earn money

$\sim p \to$ you will not work

$\sim q \to \sim p \Rightarrow$ if you will not earn money, you will not work.

3. If $\alpha, \beta \in R$ are such that $1 - 2i$ (here $i^2 = -1$) is a root of $z^2 + \alpha z + \beta = 0$, then $(\alpha - \beta)$ is equal to:

(1)7

(2) −3

(3) 3

(4)−7

Ans. (4)

Sol.

$$(1-2i)^2 + \alpha(1-2i) + \beta = 0$$

$$1 - 4 - 4i + \alpha - 2i\alpha + \beta = 0$$

$$(\alpha + \beta - 3) - i(4 + 2\alpha) = 0$$

$$\alpha + \beta - 3 = 0 \quad \& \quad 4 + 2\alpha = 0$$

$$\alpha = -2 \qquad \beta = 5$$

$$\alpha - \beta = -7$$

4. If $I_n = \int_{\pi/4}^{\pi/2} \cot^n x \, dx$, then:

(1) $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}$ are in G.P.

(2) $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}$ are in A.P.

(3) $I_2 + I_4, I_3 + I_5, I_4 + I_6$ are in A.P.

(4) $I_2 + I_4, (I_3 + I_5)^2, I_4 + I_6$ are in G.P.

Ans. (2)

Sol.

$$I_{n+2} + I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x.\operatorname{cosec}^2 x \, dx = \left[\frac{-(\cot x)^{n+1}}{n+1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$I_{n+2} + I_n = \frac{1}{n+1}$$

$$I_2 + I_4 = \frac{1}{3}, I_3 + I_5 = \frac{1}{4}, I_4 + I_6 = \frac{1}{5}$$

5. If for the matrix, $A = \begin{bmatrix} 1 & -\alpha \\ \alpha & \beta \end{bmatrix}$, $AA^T = I_2$, then the value of $\alpha^4 + \beta^4$ is:

(1) 1

(2) 3

(3) 2

(4) 4

Ans. (1)

Sol. $\begin{bmatrix} 1 & -\alpha \\ \alpha & \beta \end{bmatrix}\begin{bmatrix} 1 & \alpha \\ -\alpha & \beta \end{bmatrix} = \begin{bmatrix} 1+\alpha^2 & \alpha - \alpha\beta \\ \alpha - \alpha\beta & \alpha^2 + \beta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$1 + \alpha^2 = 1$

$\alpha^2 = 0$

$\alpha^2 + \beta^2 = 1$

$\beta^2 = 1$

$\alpha^4 = 0$

$\beta^4 = 1$

$\alpha^4 + \beta^4 = 1$

6. Let x denote the total number of one-one functions from a set A with 3 elements to a set B with 5 elements and y denote the total number of one-one functions from the set A to the set A × B. Then:

(1) $y = 273x$

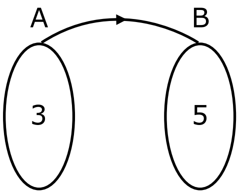
(2) $2y = 91x$

(3) $y = 91x$

(4) $2y = 273x$

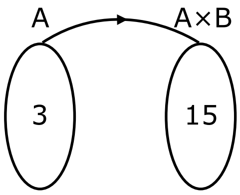
Ans. **(2)**

Sol. Number of elements in A = 3
Number of elements in B = 5
Number of elements in A × B = 15



Number of one-one function
$x = 5 \times 4 \times 3$

$x = 60$



Number of one-one function
$y = 15 \times 14 \times 13$

$y = 15 \times 4 \times \frac{14}{4} \times 13$

$y = 60 \times \frac{7}{2} \times 13$

$2y = (13)(7x)$
$2y = 91x$

7. If the curve $x^2 + 2y^2 = 2$ intersects the line $x + y = 1$ at two points P and Q, then the angle subtended by the line segment PQ at the origin is:

(1) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{4}\right)$

(2) $\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{4}\right)$

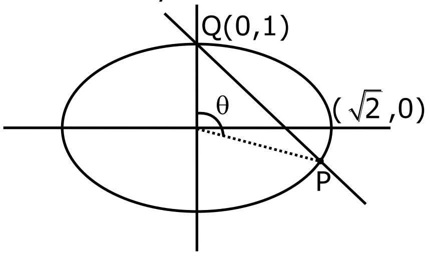
(3) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{3}\right)$

(4) $\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{3}\right)$

Ans. **(1)**

Sol. Ellipse $\frac{x^2}{2}+\frac{y^2}{1}=1$

Line $x + y = 1$



Using homogenisation

$x^2 + 2y^2 = 2(1)^2$

$x^2 + 2y^2 = 2(x + y)^2$

$x^2 + 2y^2 = 2x^2 + 2y^2 + 4xy$

$x^2 + 4xy = 0$

for $ax^2 + 2hxy + by^2 = 0$

$\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$

$\tan\theta = \left|\frac{2\sqrt{(2)^2-0}}{1+0}\right|$

$\tan\theta = -4$

$\cot\theta = -\frac{1}{4}$

$\theta = \cot^{-1}\left(-\frac{1}{4}\right)$

$\theta = \pi - \cot^{-1}\left(\frac{1}{4}\right)$

$\theta = \pi - \left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{4}\right)\right)$

$\theta = \frac{\pi}{2}+\tan^{-1}\left(\frac{1}{4}\right)$

8. The integral $\int \frac{e^{3\log_e 2x}+5e^{2\log_e 2x}}{e^{4\log_e x}+5e^{3\log_e x}-7e^{2\log_e x}}dx$, x > 0, is equal to:

(where c is a constant of integration)

(1) $\log_e|x^2 + 5x - 7| + c$

(2) $\frac{1}{4}\log_e|x^2 + 5x - 7| + c$

(3) $4\log_e|x^2 + 5x - 7| + c$

(4) $\log_e\sqrt{x^2+5x-7} + c$

Ans. (3)

Sol. $\int \frac{e^{3\log_e 2x}+5e^{2\log_e 2x}}{e^{4\log_e x}+5e^{3\log_e x}-7e^{2\log_e x}}dx$

$= \int \frac{8x^3+5(4x^2)}{x^4+5x^3-7x^2}$

$= \int \frac{8x^3+20x^2}{x^4+5x^3-7x^2}$

$= \int \frac{8x+20}{x^2+5x-7}$

$= \int \frac{4(2x+5)}{x^2+5x-7}$ $\quad \left\{\begin{array}{l}\text{Let } x^2+5x-7=t \\ (2x+5)dx = dt\end{array}\right\}$

$= \int \frac{4dt}{t}$

$= 4\ln|t|+C$

$= 4\ln|(x^2+5x-7)|+c$

9. A hyperbola passes through the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ and its transverse and conjugate axes coincide with major and minor axes of the ellipse, respectively. If the product of their eccentricities is one, then the equation of the hyperbola is:

(1) $\frac{x^2}{9}-\frac{y^2}{4}=1$

(2) $\frac{x^2}{9}-\frac{y^2}{16}=1$

(3) $x^2 - y^2 = 9$

(4) $\frac{x^2}{9}-\frac{y^2}{25}=1$

Ans. **(2)**

Sol.

$e_1=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$ foci $(\pm ae, 0)$

Foci $=(\pm 3, 0)$

Let equation of hyperbola be $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$

Passes through $(\pm 3, 0)$

$A^2=9, A=3, e_2=\frac{5}{3}$

$e_2^{\,2}=1+\frac{B^2}{A^2}$

$\frac{25}{9}=1+\frac{B^2}{9}\Rightarrow B^2=16$

Ans $\frac{x^2}{9}-\frac{y^2}{16}=1$

10. $\lim\limits_{x\to\infty}\left[\frac{1}{n}+\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+\ldots\ldots+\frac{n}{(2n-1)^2}\right]$ is equal to:

(1) 1

(2) $\frac{1}{3}$

(3) $\frac{1}{2}$

(4) $\frac{1}{4}$

Ans. **(3)**

Sol.

$$\lim_{x\to\infty}\sum_{r=0}^{n-1}\frac{n}{(n+r)^2}=\lim_{x\to\infty}\sum_{r=0}^{n-1}\frac{n^2}{n^2\left(1+\frac{r}{n}\right)^2}=\int_0^1\frac{dx}{(1+x)^2}$$

$$=-\left[\frac{1}{1+x}\right]_0^1\Rightarrow -\left[\frac{1}{2}-1\right]=\frac{1}{2}$$

11. In a group of 400 people, 160 are smokers and non-vegetarian; 100 are smokers and vegetarian and the remaining 140 are non-smokers and vegetarian. Their chances of getting a particular chest disorder are 35%, 20% and 10% respectively. A person is chosen from the group at random and is found to be suffering from the chest disorder. The probability that the selected person is a smoker and non-vegetarian is:

(1) $\frac{7}{45}$

(2) $\frac{8}{45}$

(3) $\frac{14}{45}$

(4) $\frac{28}{45}$

Ans. (4)

Sol. Based on Baye's theorem

$$\text{Probability} = \frac{\left(160 \times \frac{35}{100}\right)}{\left(160 \times \frac{35}{100}\right) + \left(100 \times \frac{20}{100}\right) + \left(140 \times \frac{10}{100}\right)}$$

$$= \frac{5600}{9000}$$

$$= \frac{28}{45}$$

12. The following system of linear equations

$3x + 3y + 2z = 9$

$3x + 2y + 2z = 9$

$x - y + 4z = 8$

(1) does not have any solution

(2) has a unique solution

(3) has a solution (α, β, γ) satisfying $\alpha + \beta^2 + \gamma^3 = 12$

(4) has infinitely many solutions

Ans. (2)

Sol. $\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 2 & 2 \\ 1 & -1 & 4 \end{vmatrix} = -20 \neq 0$ $\therefore$ unique solution

$\Delta_x = \begin{vmatrix} 9 & 3 & 2 \\ 9 & 2 & 2 \\ 8 & -1 & 4 \end{vmatrix} = 0$

$\Delta_y = \begin{vmatrix} 2 & 9 & 2 \\ 3 & 9 & 2 \\ 1 & 8 & 4 \end{vmatrix} = -20$

$\Delta_z = \begin{vmatrix} 2 & 3 & 9 \\ 3 & 2 & 9 \\ 1 & -1 & 8 \end{vmatrix} = -40$

$\therefore \quad x = \frac{\Delta_x}{\Delta} = 0$

$y = \frac{\Delta_y}{\Delta} = 1$

$z = \frac{\Delta_z}{\Delta} = 2$

Unique solution: (0, 1, 2)

13. The minimum value of $f(x) = a^{a^x} + a^{1-a^x}$, where $a, x \in R$ and $a > 0$, is equal to:

(1) $a + \frac{1}{a}$

(2) $a + 1$

(3) $2a$

(4) $2\sqrt{a}$

Ans. (4)

Sol. AM $\geq$ GM

$$\frac{a^{ax} + \frac{a}{a^{ax}}}{2} \geq \left(a^{ax} \cdot \frac{a}{a^{ax}}\right)^{1/2} \Rightarrow a^{ax} + a^{1-ax} \geq 2\sqrt{a}$$

14. A function f(x) is given by $f(x) = \frac{5^x}{5^x + 5}$, then the sum of the series

$f\left(\frac{1}{20}\right) + f\left(\frac{2}{20}\right) + f\left(\frac{3}{20}\right) + \ldots\ldots + f\left(\frac{39}{20}\right)$

is equal to:

(1) $\frac{19}{2}$

(2) $\frac{49}{2}$

(3) $\frac{39}{2}$

(4) $\frac{29}{2}$

Ans. **(3)**

Sol.

$f(x) = \frac{5^x}{5^x + 5}$ ....(i)

$f(2-x) = \frac{5^{2-x}}{5^{2-x} + 5}$

$f(2-x) = \frac{5}{5^x + 5}$......(ii)

Adding equation (i) and (ii)

$f(x) + f(2-x) = 1$

$f\left(\frac{1}{20}\right) + f\left(\frac{39}{20}\right) = 1$

$f\left(\frac{2}{20}\right) + f\left(\frac{38}{20}\right) = 1$

:

:

$f\left(\frac{19}{20}\right) + f\left(\frac{21}{20}\right) = 1$

and $f\left(\frac{20}{20}\right) = f(1) = \frac{1}{2}$

$\Rightarrow 19 + \frac{1}{2} \Rightarrow \frac{39}{2}$

15. Let α and β be the roots of $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{3a_9}$ is:

(1) 4

(2) 1

(3) 2

(4) 3

Ans. **(3)**

Sol. $x^2 - 6x - 2 = 0$ with roots α, β: $\alpha + \beta = 6$, $\alpha\beta = -2$

and $\alpha^2 - 6\alpha - 2 = 0 \Rightarrow \alpha^2 - 2 = 6\alpha$

$\beta^2 - 6\beta - 2 = 0 \Rightarrow \beta^2 - 2 = 6\beta$

Now

$$\frac{a_{10} - 2a_8}{3a_9} = \frac{\left(\alpha^{10} - \beta^{10}\right) - 2\left(\alpha^8 - \beta^8\right)}{3(\alpha^9 - \beta^9)}$$

$$= \frac{\left(\alpha^{10} - 2\alpha^8\right) - \left(\beta^{10} - 2\beta^8\right)}{3(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8\left(\alpha^2 - 2\right) - \beta^8\left(\beta^2 - 2\right)}{3(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8(6\alpha) - \beta^8(6\beta)}{3(\alpha^9 - \beta^9)} = \frac{6\left(\alpha^9 - \beta^9\right)}{3(\alpha^9 - \beta^9)} = \frac{6}{3} = 2$$

16. Let A be a 3 × 3 matrix with det(A) = 4. Let R_i denote the i^{th} row of A. If a matrix B is obtained by performing the operation $R_2 \to 2R_2 + 5R_3$ on 2A, then det(B) is equal to:

(1) 64

(2) 16

(3) 80

(4) 128

Ans. **(1)**

Sol. $A=\begin{bmatrix}R_{11} & R_{12} & R_{13}\\ R_{21} & R_{22} & R_{23}\\ R_{31} & R_{32} & R_{33}\end{bmatrix}$

$2A=\begin{bmatrix}2R_{11} & 2R_{12} & 2R_{13}\\ 2R_{21} & 2R_{22} & 2R_{23}\\ 2R_{31} & 2R_{32} & 2R_{33}\end{bmatrix}$

$R_2 \to 2R_2 + 5R_3$

$B=\begin{bmatrix}2R_{11} & 2R_{12} & 2R_{13}\\ 4R_{21}+10R_{31} & 4R_{22}+10R_{32} & 4R_{23}+10R_{33}\\ 2R_{31} & 2R_{32} & 2R_{33}\end{bmatrix}$

$R_2 \to R_2 - 5R_3$

$B=\begin{bmatrix}2R_{11} & 2R_{12} & 2R_{13}\\ 4R_{21} & 4R_{22} & 4R_{23}\\ 2R_{31} & 2R_{32} & 2R_{33}\end{bmatrix}$

$|B|=\begin{vmatrix}2R_{11} & 2R_{12} & 2R_{13}\\ 4R_{21} & 4R_{22} & 4R_{23}\\ 2R_{31} & 2R_{32} & 2R_{33}\end{vmatrix}$

$|B|=2\times 2\times 4\begin{vmatrix}R_{11} & R_{12} & R_{13}\\ R_{21} & R_{22} & R_{23}\\ R_{31} & R_{32} & R_{33}\end{vmatrix}$

$=16\times 4$

$=64$

17. The shortest distance between the line $x - y = 1$ and the curve $x^2 = 2y$ is:

(1) $\frac{1}{2}$

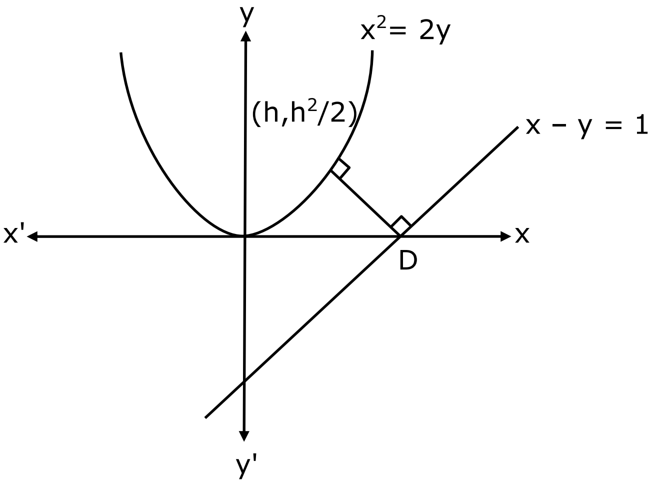
(2) 0

(3) $\frac{1}{2\sqrt{2}}$

(4) $\frac{1}{\sqrt{2}}$

Ans. **(3)**

Sol. Shortest distance must be along common normal



m_1 (slope of line $x-y = 1$) = 1 $\Rightarrow$ slope of perpendicular line $=-1$

$m_2 = \frac{2x}{2} = x \Rightarrow m_2 = h \Rightarrow$ slope of normal $-\frac{1}{h}$

$-\frac{1}{h} = -1 \Rightarrow h = 1$

so point is $\left(1, \frac{1}{2}\right)$

$D = \left|\frac{1-\frac{1}{2}-1}{\sqrt{1+1}}\right| = \frac{1}{2\sqrt{2}}$

18. Let A be a set of all 4-digit natural numbers whose exactly one digit is 7. Then the probability that a randomly chosen element of A leaves remainder 2 when divided by 5 is:

(1) $\frac{1}{5}$

(2) $\frac{2}{9}$

(3) $\frac{97}{297}$

(4) $\frac{122}{297}$

Ans. **(3)**

Sol. Total cases

$(4 \times 9 \times 9 \times 9) - (3 \times 9 \times 9)$

Probability $= \frac{(3 \times 9 \times 9) - (2 \times 9) + (8 \times 9 \times 9)}{(4 \times 9^3) - (3 \times 9^2)}$

$= \frac{97}{217}$

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19. $\operatorname{cosec}\left[2\cot^{-1}(5)+\cos^{-1}\left(\frac{4}{5}\right)\right]$ is equal to:

(1) $\frac{75}{56}$

(2) $\frac{65}{56}$

(3) $\frac{56}{33}$

(4) $\frac{65}{33}$

Ans. **(2)**

Sol. $\operatorname{cosec}\left(2\cot^{-1}(5)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

$\operatorname{cosec}\left(2\tan^{-1}\left(\frac{1}{5}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

$=\operatorname{cosec}\left(\tan^{-1}\left(\frac{2\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

$=\operatorname{cosec}\left(\tan^{-1}\left(\frac{5}{12}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

Let $\tan^{-1}(5/12)=\theta \Rightarrow \sin\theta=\frac{5}{13}$, $\cos\theta=\frac{12}{13}$

and $\cos^{-1}\left(\frac{4}{5}\right)=\phi \Rightarrow \cos\phi=\frac{4}{5}$ and $\sin\phi=\frac{3}{5}$

$=\operatorname{cosec}(\theta+\phi)$

$=\frac{1}{\sin\theta\cos\phi+\cos\theta\sin\phi}$

$=\frac{1}{\frac{5}{13}.\frac{4}{5}+\frac{12}{13}.\frac{3}{5}}=\frac{65}{56}$

20. If $0 < x, y < \pi$ and $\cos x + \cos y - \cos(x + y) = \frac{3}{2}$, then $\sin x + \cos y$ is equal to:

(1) $\frac{1+\sqrt{3}}{2}$

(2) $\frac{1-\sqrt{3}}{2}$

(3) $\frac{\sqrt{3}}{2}$

(4) $\frac{1}{2}$

Ans. (1)

Sol.

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)-\left[2\cos^2\left(\frac{x+y}{2}\right)-1\right]=\frac{3}{2}$$

$$2\cos\left(\frac{x+y}{2}\right)\left[\cos\left(\frac{x-y}{2}\right)-\cos\left(\frac{x+y}{2}\right)\right]=\frac{1}{2}$$

$$2\cos\left(\frac{x+y}{2}\right)\left[2\sin\left(\frac{x}{2}\right).\sin\left(\frac{y}{2}\right)\right]=\frac{1}{2}$$

$$\cos\left(\frac{x+y}{2}\right).\sin\left(\frac{x}{2}\right).\sin\left(\frac{y}{2}\right)=\frac{1}{8}$$

Possible when $\frac{x}{2}=30^\circ$ & $\frac{y}{2}=30^\circ$

$x=y=60^\circ$

$$\sin x+\cos y=\frac{\sqrt{3}}{2}+\frac{1}{2}=\frac{\sqrt{3}+1}{2}$$

SECTION-B

1. If $\lim_{x\to 0}\frac{ax-\left(e^{4x}-1\right)}{ax\left(e^{4x}-1\right)}$ exists and is equal to b, then the value of a – 2b is ______.

Ans. **(5)**

$$\lim_{x\to 0}\frac{ax-\left(e^{4x}-1\right)}{ax\left(e^{4x}-1\right)}$$

Applying L' Hospital Rule

$$\lim_{x\to 0}\frac{a-4e^{4x}}{a\left(e^{4x}-1\right)+ax\left(4e^{4x}\right)} \quad \text{So } a=4$$

Sol. Applying L' Hospital Rule

$$\lim_{x\to 0}\frac{-16e^{4x}}{a\left(4e^{4x}\right)+a\left(4e^{4x}\right)+ax\left(16e^{4x}\right)}$$

$$\frac{-16}{4a+4a}=\frac{-16}{32}=-\frac{1}{2}=b$$

$$a-2b=4-2\left(\frac{-1}{2}\right)=4+1=5$$

2. A line is a common tangent to the circle $(x-3)^2+y^2=9$ and the parabola $y^2=4x$. If the two points of contact (a, b) and (c, d) are distinct and lie in the first quadrant, then 2(a+c) is equal to ______.

Ans. **(9)**

Sol. Circle: $(x-3)^2+y^2=9$

Parabola: $y^2=4x$

Let tangent $y=mx+\frac{a}{m}$

$$y=mx+\frac{1}{m}$$

$m^2x-my+1=0$

the above line is also tangent to circle

$(x-3)^2+y^2=9$

$\therefore \perp$ from (3, 0) = 3

$$\left|\frac{3m^2-0+1}{\sqrt{m^2+m^4}}\right|=3$$

$(3m^2+1)^2=9(m^2+m^4)$

$6m^2 + 1 + 9m^4 = 9m^2 + 9m^4$

$3m^2 = 1$

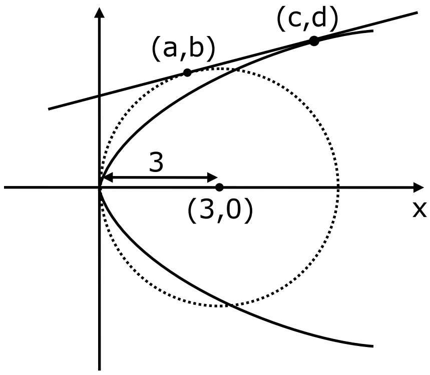
$m = \pm\frac{1}{\sqrt{3}}$

$\therefore$tangent is

$y = \frac{1}{\sqrt{3}}x + \sqrt{3}$ or $y = -\frac{1}{\sqrt{3}}x - \sqrt{3}$

(it will be used) (rejected)

$m = \frac{1}{\sqrt{3}}$



for Parabola $\left(\frac{a}{m^2}, \frac{2a}{m}\right) \equiv (3, 2\sqrt{3})$

(c, d)

for Circle $y = \frac{1}{\sqrt{3}}x + \sqrt{3}$ & $(x - 3)^2 + y^2 = 9$

solving, $(x - 3)^2 + \left(\frac{1}{\sqrt{3}}x + \sqrt{3}\right)^2 = 9$

$x^2 + 9 - 6x + \frac{1}{3}x^2 + 3 + 2x = 9$

$\frac{4}{3}x^2 - 4x + 3 = 0$

$4x^2 - 12x + 9 = 0$

$4x^2 - 6x - 6x + 9 = 0$

$2x(2x - 3) - 3(2x - 3) = 0$

$(2x - 3)(2x - 3) = 0$

$x = \frac{3}{2}$

$\therefore \quad y = \frac{1}{\sqrt{3}}\left(\frac{3}{2}\right) + \sqrt{3}$

$y = \frac{\sqrt{3}}{2} + \sqrt{3}$

$y = \frac{3\sqrt{3}}{2}$

$(a, b) \equiv \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

$2(a + c) = 2\left(\frac{3}{2} + 3\right)$

$= 2\left(\frac{3}{2} + \frac{6}{2}\right)$

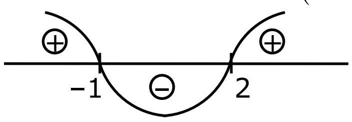
$= 9$

3. The value of $\int_{-2}^{2} \left|3x^2 - 3x - 6\right| dx$ is ______.

Ans. (19)

Sol. $3\int_{-2}^{2} \left|x^2 - x - 2\right| dx \qquad x^2 - x - 2$

$= (x-2)(x+1)$



$= 3\left\{\int_{-2}^{-1} (x^2 - x - 2)\,dx + \int_{-1}^{2} (-x^2 + x + 2)\,dx\right\}$

$= 3\left[\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)_{-2}^{-1} - \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)_{-1}^{2}\right]$

$= 19$

4. If the remainder when x is divided by 4 is 3, then the remainder when $(2020+x)^{2022}$is divided by 8 is ______.

Ans. **(1)**

Sol. Let x = 4k + 3

$(2020 + x)^{2022}$

$= (2020 + 4k + 3)^{2022}$

$= (4(505) + 4k + 3)^{2022}$

$= (4P + 3)^{2022}$

$= (4P + 4 - 1)^{2022}$

$= (4A - 1)^{2022}$

$^{2022}C_0(4A)^0(-1)^{2022} + {}^{2022}C_1(4A)^1(-1)^{2021} + \ldots\ldots$

$1 + 8\lambda$

Reminder is 1.

5. A line 'ℓ' passing through origin is perpendicular to the lines

$$\ell_1 : \vec{r} = (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}$$

$$\ell_2 : \vec{r} = (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}$$

If the co-ordinates of the point in the first octant on 'ℓ_2' at the distance of $\sqrt{17}$ from the point of intersection of 'ℓ' and 'ℓ_1' are (a, b, c), then 18(a+b+c) is equal to ______.

Ans. (44)

Sol. $\ell_1 : \vec{r} = (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}$

$\ell_1 : \frac{x-3}{1} = \frac{y+1}{2} = \frac{z-4}{2} \Rightarrow$ D.R. of ℓ_1 = 1, 2, 2

$\ell_2 : \vec{r} = (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}$

$\ell_2 : \frac{x-3}{2} = \frac{y-3}{2} = \frac{z-2}{1} \Rightarrow$ D.R. of ℓ_2 = 2, 2, 1

D.R. of ℓ is $\perp$ to $\ell_1 \& \ell_2$

$\therefore$ D.R. of ℓ || $(\ell_1 \times \ell_2) \Rightarrow \langle -2, 3, -2 \rangle$

$\therefore$ Equation of $\ell : \frac{x}{2} = \frac{y}{-3} = \frac{z}{2}$

Solving ℓ & ℓ_1

$(2\lambda, -3\lambda, 2\lambda) = (\mu + 3, 2\mu - 1, 2\mu + \mu)$

$\Rightarrow\ 2\lambda = \mu + 3$

$-3\lambda = 2\mu - 1$

$2\lambda = 2\mu + 4$

$\Rightarrow\ \mu + 3 = 2\mu + 4$

$\mu = -1$

$\lambda = 1$

P(2, −3, 2) {intersection point}

Let, $Q(2\nu + 3, 2\nu + 3, \nu + 2)$ be point on ℓ_2

Now, $PQ = \sqrt{(2\nu+3-2)^2+(2\nu+3+3)^2+(\nu+2-2)^2} = \sqrt{17}$

$\Rightarrow\ (2\nu + 1)^2 + (2\nu + 6)^2 + (\nu)^2 = 17$

$\Rightarrow\ 9\nu^2 + 28\nu + 36 + 1 - 17 = 0$

$\Rightarrow\ 9\nu^2 + 28\nu + 20 = 0$

$\Rightarrow\ 9\nu^2 + 18\nu + 10\nu + 20 = 0$

$\Rightarrow\ (9\nu + 10)(\nu + 2) = 0$

$\Rightarrow\ \nu = -2$ (rejected), $-\frac{10}{9}$ (accepted)

$Q\left(3-\frac{20}{9},\ 3-\frac{20}{9},\ 2-\frac{10}{9}\right)$

$\left(\frac{7}{9},\ \frac{7}{9},\ \frac{8}{9}\right)$

$\therefore\ 18(a + b + c)$

$= 18\left(\frac{7}{9}+\frac{7}{9}+\frac{8}{9}\right)$

$= 44$

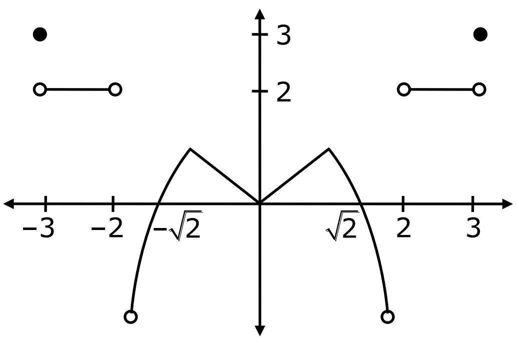
6. A function f is defined on $[-3,3]$ as

$$f(x)=\begin{cases}\min\{|x|, 2-x^2\}, & -2\le x\le 2\\ [|x|] & ,2<|x|\le 3\end{cases}$$

where $[x]$ denotes the greatest integer $\le x$. The number of points, where f is not differentiable in $(-3,3)$ is _______.

Ans. **(5)**

Sol.



Points of non-differentiability in $(-3, 3)$ are at $x = -2, -1, 0, 1, 2$.

i.e. 5 points.

7. If the curves $x = y^4$ and $xy = k$ cut at right angles, then $(4k)^6$ is equal to _______.

Ans. **4**

Sol. $4y^3\frac{dy}{dx}=1$ & $x\frac{dy}{dx}+y=0$

$m_1 = \frac{1}{4y^3}$ $\quad \frac{dy}{dx}=\frac{-y}{x}=m_2$

$m_1m_2 = -1$

$\frac{1}{4.y^3}\times\frac{-y}{x}=-1 \quad \because x=y^4$

$\frac{1}{4.y^6}=1$ and $xy=k$

$y^6=\frac{1}{4}$ $\quad \Rightarrow k = y^5$

$\Rightarrow k^6 = y^{30}$

$\Rightarrow k^6=\left(\frac{1}{4}\right)^5$

$\therefore (4k)^6 = 4^6\times k^6 = 4$

8. The total number of two digit numbers 'n', such that 3^n+7^n is a multiple of 10, is _______.

Ans. **(45)**

Sol. $\because 7^n = (10-3)^n = 10K + (-3)^n$

$\therefore 7^n + 3^n = 10K + (-3)^n + 3^n$ → 10K if n = odd

→ $10K + 2.3^n$ if n = even

Let n = 2t; $t \in N$

$\therefore 3^n = 3^{2t} = (10-1)^t$

$= 10p + (-1)^t$

$= 10p \pm 1$

$\therefore$ if n = even then $7^n + 3^n$ will not be multiply of 10

So if n is odd then only $7^n + 3^n$ will be multiply of 10

$\therefore n = 11, 13, 15, ..., 99$

$\therefore$ Ans 45

9. Let $\vec{a} = \hat{i} + \alpha\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - \alpha\hat{j} + \hat{k}$. If the area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $8\sqrt{3}$ square units, then $\vec{a}\cdot\vec{b}$ is equal to _______.

Ans. **(2)**

Sol. $\vec{a} = \hat{i} + \alpha\hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} - \alpha\hat{j} + \hat{k}$

Area of parallelogram = $|\vec{a} \times \vec{b}|$

$= \left|(\hat{i} + \alpha\hat{j} + 3\hat{k}) \times (3\hat{i} - \alpha\hat{j} + \hat{k})\right|$

$8\sqrt{3} = \left|(4\alpha)\hat{i} + 8\hat{j} - (4\alpha)\hat{k}\right|$

$(64)(3) = 16\alpha^2 + 64 + 16\alpha^2$

$(64)(3) = 32\alpha^2 + 64$

$6 = \alpha^2 + 2$

$\alpha^2 = 4$

$\therefore$ $\vec{a} = \hat{i} + \alpha\hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} - \alpha\hat{j} + \hat{k}$

$\vec{a}\cdot\vec{b} = 3 - \alpha^2 + 3$

$= 6 - \alpha^2$

$= 6 - 4$

$= 2$

10. If the curve $y = y(x)$ represented by the solution of the differential equation $(2xy^2 - y)dx + xdx = 0$, passes through the intersection of the lines, $2x - 3y=1$ and $3x+2y=8$, then $|y(1)|$ is equal to ______.

Ans. 1

Sol. Given,

$(2xy^2 - y)dx + xdx = 0$

$\Rightarrow \dfrac{dy}{dx} + 2y^2 - \dfrac{y}{x} = 0$

$\Rightarrow -\dfrac{1}{y^2}\dfrac{dy}{dx} + \dfrac{1}{y}\left(\dfrac{1}{x}\right) = 2$

$\dfrac{1}{y} = z$

$-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\Rightarrow \dfrac{dz}{dx} + z\left(\dfrac{1}{x}\right) = 2$

$\text{I.F.} = e^{\int \frac{1}{x}dx} = x$

$\therefore \; z(x) = \int 2(x)\,dx = x^2 + c$

$\Rightarrow \dfrac{x}{y} = x^2 + c$

As it passes through P(2, 1)

[Point of intersection of $2x - 3y = 1$ and $3x + 2y = 8$]

$\therefore \dfrac{2}{1} = 4 + c$

$\Rightarrow c = -2$

$\Rightarrow \dfrac{x}{y} = x^2 - 2$

Put $x = 1$

$\dfrac{1}{y} = 1 - 2 = -1$

$\Rightarrow y(1) = -1$

$\Rightarrow |y(1)| = 1$