

**Learning Temple**

**IIT/NEET ACADEMY**

**26<sup>th</sup> Feb. 2021 | Shift - 2**  
**MATHEMATICS**

1. Let L be a line obtained from the intersection of two planes  $x + 2y + z = 6$  and  $y + 2z = 4$ . If point  $P(\alpha, \beta, \gamma)$  is the foot of perpendicular from  $(3, 2, 1)$  on L, then the value of  $21(\alpha + \beta + \gamma)$  equals :

- (1) 142  
 (2) 68  
 (3) 136  
 (4) 102

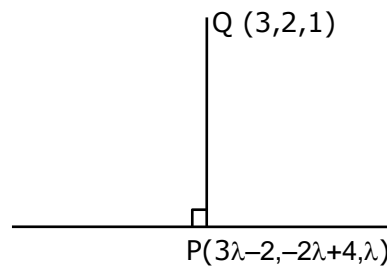
**Ans. (4)**

Sol. Dr's of line  $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 3\hat{i} - 2\hat{j} + \hat{k}$

Dr/s :-  $(3, -2, 1)$

Points on the line  $(-2, 4, 0)$

Equation of the line  $\frac{x+2}{3} = \frac{y-4}{-2} = \frac{z}{1} = \lambda$



Dr's of PQ ;  $3\lambda - 5, -2\lambda + 2, \lambda - 1$

Dr's of y lines are  $(3, -2, 1)$

Since  $PQ \perp$  line

$$3(3\lambda - 5) - 2(-2\lambda + 2) + 1(\lambda - 1) = 0$$

$$\lambda = \frac{10}{7}$$

$$P\left(\frac{16}{7}, \frac{8}{7}, \frac{10}{7}\right)$$

$$21(\alpha + \beta + \gamma) = 21\left(\frac{34}{7}\right) = 102$$

2. The sum of the series  $\sum_{n=1}^{\infty} \frac{n^2 + 6n + 10}{(2n+1)!}$  is equal to :

(1)  $\frac{41}{8} e + \frac{19}{8} e^{-1} - 10$

(2)  $-\frac{41}{8} e + \frac{19}{8} e^{-1} - 10$

(3)  $\frac{41}{8} e - \frac{19}{8} e^{-1} - 10$

(4)  $\frac{41}{8} e + \frac{19}{8} e^{-1} + 10$

**Ans. (3)**

Sol.  $\sum_{n=1}^{\infty} \frac{n^2 + 6n + 10}{(2n+1)!}$

Put  $2n + 1 = r$ , where  $r = 3, 5, 7, \dots$

$$\Rightarrow n = \frac{r-1}{2}$$

$$\frac{n^2 - 6n + 10}{(2n+1)!} = \frac{\left(\frac{r-1}{2}\right)^2 + 3r - 3 + 10}{r!} = \frac{r^2 + 10r + 29}{4r!}$$

$$\text{Now } \sum_{r=3,5,7} \frac{r(r-1) + 11r + 29}{4r!} = \frac{1}{4} \sum_{r=3,5,7,\dots} \left( \frac{1}{(r-2)!} + \frac{11}{(r-1)!} + \frac{29}{r!} \right)$$

$$= \frac{1}{4} \left\{ \left( \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots \right) + 11 \left( \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots \right) + 29 \left( \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \dots \right) \right\}$$

$$= \frac{1}{4} \left\{ \frac{e - \frac{1}{e}}{2} + 11 \left( \frac{e + \frac{1}{e} - 2}{2} \right) + 29 \left( \frac{e - \frac{1}{e} - 2}{2} \right) \right\}$$

$$= \frac{1}{8} \left\{ e - \frac{1}{e} + 11e + \frac{11}{e} - 22 + 29e - \frac{29}{e} - 58 \right\}$$

$$= \frac{1}{8} \left\{ 41e - \frac{19}{e} - 80 \right\}$$

3. Let  $f(x)$  be a differentiable function at  $x = a$  with  $f'(a) = 2$  and  $f(a) = 4$ . Then  $\lim_{x \rightarrow a} \frac{xf(a) - af(x)}{x - a}$  equals :

(1)  $2a + 4$

(2)  $2a - 4$

(3)  $4 - 2a$

(4)  $a + 4$

**Ans. (3)**

**Sol.** By L-H rule

$$L = \lim_{x \rightarrow a} \frac{f(a) - af'(x)}{1}$$

$$\therefore L = 4 - 2a$$

**4.** Let A (1, 4) and B(1, -5) be two points. Let P be a point on the circle  $(x - 1)^2 + (y - 1)^2 = 1$  such that  $(PA)^2 + (PB)^2$  have maximum value, then the points, P, A and B lie on :

(1) a parabola

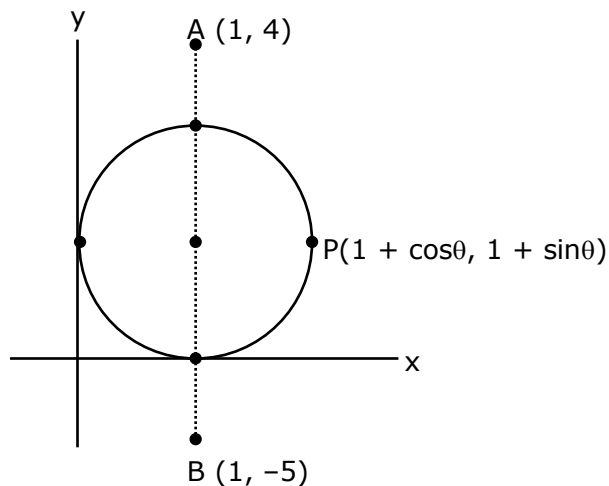
(2) a straight line

(3) a hyperbola

(4) an ellipse

**Ans. (2)**

**Sol.**



$$\therefore PA^2 = \cos^2\theta + (\sin\theta - 3)^2 = 10 - 6 \sin\theta$$

$$PB^2 = \cos^2\theta + (\sin\theta - 6)^2 = 37 - 12 \sin\theta$$

$$PA^2 + PB^2 \Big|_{\max.} = 47 - 18 \sin\theta \Big|_{\min.} \Rightarrow \theta = \frac{3\pi}{2}$$

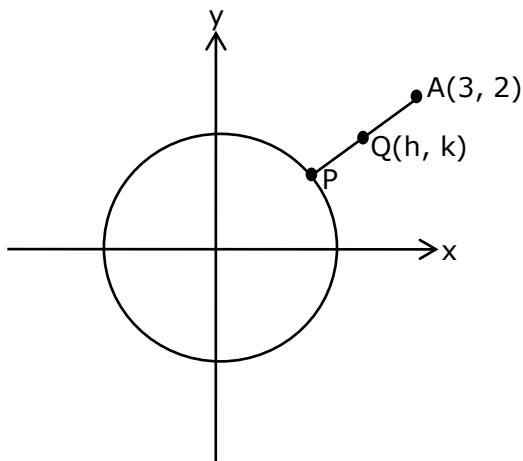
$\therefore$  P, A, B lie on a line  $x = 1$

5. If the locus of the mid-point of the line segment from the point (3, 2) to a point on the circle,  $x^2 + y^2 = 1$  is a circle of the radius  $r$ , then  $r$  is equal to :

- (1)  $\frac{1}{4}$   
 (2)  $\frac{1}{2}$   
 (3) 1  
 (4)  $\frac{1}{3}$

**Ans. (2)**

Sol.



$$\therefore P \equiv (2h - 3, 2k - 2) \rightarrow \text{on circle}$$

$$\therefore \left(h - \frac{3}{2}\right)^2 + (k - 1)^2 = \frac{1}{4}$$

$$\Rightarrow \text{radius} = \frac{1}{2}$$

6. Let slope of the tangent line to a curve at any point  $P(x, y)$  be given by  $\frac{xy^2 + y}{x}$ . If the curve intersects the line  $x + 2y = 4$  at  $x = -2$ , then the value of  $y$ , for which the point  $(3, y)$  lies on the curve, is :

- (1)  $-\frac{18}{11}$   
 (2)  $-\frac{18}{19}$   
 (3)  $-\frac{4}{3}$

(4)  $\frac{18}{35}$

**Ans. (2)**

Sol.  $\frac{dy}{dx} = \frac{xy^2 + y}{x}$   
 $\Rightarrow \frac{xdy - ydx}{y^2} = xdx$

$\Rightarrow -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$

$\Rightarrow \frac{-x}{y} = \frac{x^2}{2} + C$

Curve intersect the line  $x + 2y = 4$  at  $x = -2$

So,  $-2 + 2y = 4 \Rightarrow y = 3$

So the curve passes through  $(-2, 3)$

$\Rightarrow \frac{2}{3} = 2 + C$

$\Rightarrow C = \frac{-4}{3}$

$\therefore$  curve is  $\frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$

It also passes through  $(3, y)$

$\frac{-3}{y} = \frac{9}{2} - \frac{4}{3}$

$\Rightarrow \frac{-3}{y} = \frac{19}{6}$

$\Rightarrow y = -\frac{18}{19}$

**7.** Let  $A_1$  be the area of the region bounded by the curves  $y = \sin x$ ,  $y = \cos x$  and  $y$ -axis in the first quadrant. Also, let  $A_2$  be the area of the region bounded by the curves  $y = \sin x$ ,  $y = \cos x$ ,  $x$ -axis and  $x = \frac{\pi}{2}$  in the first quadrant. Then,

(1)  $A_1 = A_2$  and  $A_1 + A_2 = \sqrt{2}$

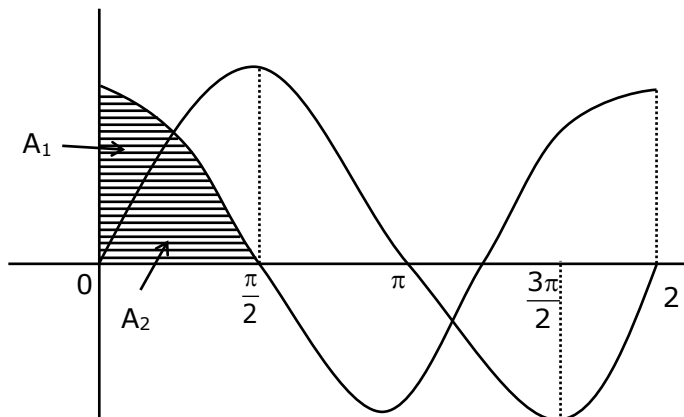
(2)  $A_1 : A_2 = 1 : 2$  and  $A_1 + A_2 = 1$

(3)  $2A_1 = A_2$  and  $A_1 + A_2 = 1 + \sqrt{2}$

(4)  $A_1 : A_2 = 1 : \sqrt{2}$  and  $A_1 + A_2 = 1$

**Ans. (4)**

Sol.  $A_1 + A_2 = \int_0^{\pi/2} \cos x \cdot dx = \sin x \Big|_0^{\pi/2} = 1$



$$A_1 = \int_0^{\pi/4} (\cos x - \sin x) dx = (\sin x + \cos x) \Big|_0^{\pi/4} = \sqrt{2} - 1$$

$$\therefore A_2 = 1 - (\sqrt{2} - 1) = 2 - \sqrt{2}$$

$$\therefore \frac{A_1}{A_2} = \frac{\sqrt{2} - 1}{\sqrt{2}(\sqrt{2} - 1)} = \frac{1}{\sqrt{2}}$$

**8.** If  $0 < a, b < 1$ , and  $\tan^{-1} a + \tan^{-1} b = \frac{\pi}{4}$ , then the value of

$$(a + b) - \left(\frac{a^2 + b^2}{2}\right) + \left(\frac{a^3 + b^3}{3}\right) - \left(\frac{a^4 + b^4}{4}\right) + \dots \text{ is :}$$

(1)  $\log_e 2$

(2)  $\log_e \left(\frac{e}{2}\right)$

(3)  $e$

(4)  $e^2 - 1$

**Ans. (1)**

Sol.  $\tan^{-1} \left(\frac{a+b}{1-ab}\right) = \frac{\pi}{4} \Rightarrow a + b = 1 - ab \Rightarrow (1 + a)(1 + b) = 2$

Now,  $(a + b) - \left(\frac{a^2 + b^2}{2}\right) + \left(\frac{a^3 + b^3}{3}\right) \dots \infty$

$$= \left(a - \frac{a^2}{2} + \frac{a^3}{3} \dots\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} \dots\right)$$

$$\log_e (1 + a) + \log_e (1 + b) = \log_e (1 + a) (1 + b) = \log_e 2$$

9. Let  $F_1(A, B, C) = (A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$  and  $F_2(A, B) = (A \vee B) \vee (B \rightarrow \sim A)$  be two logical expressions. Then :
- (1)  $F_1$  is not a tautology but  $F_2$  is a tautology
  - (2)  $F_1$  is a tautology but  $F_2$  is not a tautology
  - (3)  $F_1$  and  $F_2$  both are tautologies
  - (4) Both  $F_1$  and  $F_2$  are not tautologies

**Ans. (1)**

Sol. Truth table for  $F_1$

| A | B | C | $\sim A$ | $\sim B$ | $\sim C$ | $A \vee \sim B$ | $A \vee B$ | $\sim C \vee (A \vee B)$ | $[\sim C \wedge (A \vee B)] \vee \sim A$ | $(A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$ |
|---|---|---|----------|----------|----------|-----------------|------------|--------------------------|--|---|
| T | T | T | F        | F        | F        | F               | T          | F                        | F  | F   |
| T | T | F | F        | F        | T        | F               | T          | T                        | T  | T   |
| T | F | T | F        | T        | F        | T               | T          | F                        | F  | T   |
| T | F | F | F        | T        | T        | T               | T          | T                        | T  | T   |
| F | T | T | T        | F        | F        | F               | T          | F                        | T  | T   |
| F | T | F | T        | F        | T        | F               | T          | T                        | T  | T   |
| F | F | T | T        | T        | F        | F               | F          | F                        | T  | T   |
| F | F | F | T        | T        | T        | F               | F          | F                        | T  | T   |

Not a tautology

Truth table for  $F_2$

| A | B | $A \vee B$ | $\sim A$ | $B \rightarrow \sim A$ | $(A \vee B) \vee (B \rightarrow \sim A)$ |
|---|---|------------|----------|------------------------|--|
| T | T | T          | F        | F                      | T  |
| T | F | T          | F        | T                      | T  |
| F | T | T          | T        | T                      | T  |
| F | F | F          | T        | T                      | T  |

$F_1$  not shows tautology and  $F_2$  shows tautology

10. Consider the following system of equations :

$$x + 2y - 3z = a$$

$$2x + 6y - 11z = b$$

$$x - 2y + 7z = c,$$

Where a, b and c are real constants. Then the system of equations :

- (1) has a unique solution when  $5a = 2b + c$
- (2) has infinite number of solutions when  $5a = 2b + c$
- (3) has no solution for all a, b and c
- (4) has a unique solution for all a, b and c

**Ans. (2)**

Sol. 
$$D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$$

$$= 20 - 2(25) - 3(-10)$$

$$= 20 - 50 + 30 = 0$$

$$D_1 = \begin{vmatrix} a & 2 & -3 \\ b & 6 & -11 \\ c & -2 & 7 \end{vmatrix}$$



$$= 20a - 2(7b + 11c) - 3(-2b - 6c)$$

$$= 20a - 14b - 22c + 6b + 18c$$

$$= 20a - 8b - 4c$$

$$= 4(5a - 2b - c)$$

$$D_2 = \begin{vmatrix} 1 & a & -3 \\ 2 & b & -11 \\ 1 & c & 7 \end{vmatrix}$$

$$= 7b + 11c - a(25) - 3(2c - b)$$

$$= 7b + 11c - 25a - 6c + 3b$$

$$= -25a + 10b + 5c$$

$$= -5(5a - 2b - c)$$

$$D_3 = \begin{vmatrix} 1 & 2 & a \\ 2 & 6 & b \\ 1 & -2 & c \end{vmatrix}$$

$$= 6c + 2b - 2(2c - b) - 10a$$

$$= -10a + 4b + 2c$$

$$= -2(5a - 2b - c)$$

for infinite solution

$$D = D_1 = D_2 = D_3 = 0$$

$$\Rightarrow 5a = 2b + c$$

- 11.** A seven digit number is formed using digit 3, 3, 4, 4, 4, 5, 5. The probability, that number so formed is divisible by 2, is :

(1)  $\frac{6}{7}$

(2)  $\frac{4}{7}$

(3)  $\frac{3}{7}$

(4)  $\frac{1}{7}$

**Ans. (3)**

Sol.  $n(S) = \frac{7!}{2!3!2!}$

$$n(E) = \frac{6!}{2!2!2!}$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{6!}{7!} \times \frac{2!3!2!}{2!2!2!}$$

$$\frac{1}{7} \times 3 = \frac{3}{7}$$

**12.** If vectors  $\vec{a}_1 = x\hat{i} - \hat{j} + \hat{k}$  and  $\vec{a}_2 = \hat{i} + y\hat{j} + z\hat{k}$  are collinear, then a possible unit vector parallel to the vector  $x\hat{i} + y\hat{j} + z\hat{k}$  is :

(1)  $\frac{1}{\sqrt{2}} (-\hat{j} + \hat{k})$

(2)  $\frac{1}{\sqrt{2}} (\hat{i} - \hat{j})$

(3)  $\frac{1}{\sqrt{3}} (\hat{i} - \hat{j} + \hat{k})$

(4)  $\frac{1}{\sqrt{3}} (\hat{i} + \hat{j} - \hat{k})$

**Ans. (3)**

Sol.  $\frac{x}{1} = -\frac{1}{y} = \frac{1}{z} = \lambda$  (let)

Unit vector parallel to  $x\hat{i} + y\hat{j} + z\hat{k} = \pm \frac{\left(\lambda\hat{i} - \frac{1}{\lambda}\hat{j} + \frac{1}{\lambda}\hat{k}\right)}{\sqrt{\lambda^2 + \frac{2}{\lambda^2}}}$

For  $\lambda = 1$ , it is  $\pm \frac{(\hat{i} - \hat{j} + \hat{k})}{\sqrt{3}}$

**13.** For  $x > 0$ , if  $f(x) = \int_1^x \frac{\log_e t}{1+t} dt$ , then  $f(e) + f\left(\frac{1}{e}\right)$  is equal to :

(1)  $\frac{1}{2}$

(2) -1

(3) 1

(4) 0

**Ans. (1)**

Sol.  $f(e) + f\left(\frac{1}{e}\right) = \int_1^e \frac{\log_e t}{1+t} dt + \int_1^{1/e} \frac{\log_e t}{1+t} dt = I_1 + I_2$

$I_2 = \int_1^{1/e} \frac{\log_e t}{1+t} dt$  put  $t = \frac{1}{z}, dt = -\frac{dz}{z^2}$

$= \int_1^e -\frac{\log_e \frac{1}{z}}{1 + \frac{1}{z}} \times \left(-\frac{dz}{z^2}\right) = \int_1^e \frac{\log_e z}{z(z+1)} dz$

$$\begin{aligned}
 f(e) + f\left(\frac{1}{e}\right) &= \int_1^e \frac{t \ln t}{1+t} dt + \int_1^e \frac{t \ln t}{t(t+1)} dt = \int_1^e \frac{t \ln t}{1+t} + \frac{t \ln t}{t(t+1)} dt \\
 &= \int_1^e \frac{t \ln t}{t} dt \quad \left\{ \ln t = u, \frac{1}{t} dt \right\} \\
 &= du = \int_0^1 u du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2}
 \end{aligned}$$

**14.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), & \text{if } x < -1 \\ |ax^2 + x + b|, & \text{if } -1 \leq x \leq 1 \\ \sin(\pi x) & \text{if } x > 1 \end{cases}$

If  $f(x)$  is continuous on  $\mathbb{R}$ , then  $a + b$  equals :

- (1) 3
- (2) -1
- (3) -3
- (4) 1

**Ans. (2)**

**Sol.** If  $f$  is continuous at  $x = -1$ , then  $f(-1^-) = f(-1)$

$$\Rightarrow 2 = |a - 1 + b|$$

$$\Rightarrow |a + b - 1| = 2 \dots (i)$$

similarly

$$f(1^-) = f(1)$$

$$\Rightarrow |a + b + 1| = 0$$

$$\Rightarrow a + b = -1$$

**15.** Let  $A = \{1, 2, 3, \dots, 10\}$  and  $f: A \rightarrow A$  be defined as

$$f(k) = \begin{cases} k+1 & \text{if } k \text{ is odd} \\ k & \text{if } k \text{ is even} \end{cases} \quad \text{Then the number of possible functions}$$

$g : A \rightarrow A$  such that  $\text{gof} = f$  is :

- (1)  $10^5$
- (2)  ${}^{10}C_5$
- (3)  $5^5$
- (4)  $5!$

**Ans. (1)**

Sol.  $g(f(x)) = f(x)$

$\Rightarrow g(x) = x$ , when  $x$  is even.

5 elements in  $A$  can be mapped to any 10

So,  $10^5 \times 1 = 10^5$

**16.** A natural number has prime factorization given by  $n = 2^x 3^y 5^z$ , where  $y$  and  $z$  are such that  $y + z = 5$  and  $y^{-1} + z^{-1} = \frac{5}{6}$ ,  $y > z$ . Then the number of odd divisors of  $n$ , including 1, is :

(1) 11

(2) 6x

(3) 12

(4) 6

**Ans. (3)**

Sol.  $y + z = 5$  ... (1)

$$\frac{1}{y} + \frac{1}{z} = \frac{5}{6}$$

$$\Rightarrow \frac{y+z}{yz} = \frac{5}{6}$$

$$\Rightarrow \frac{5}{yz} = \frac{5}{6}$$

$$\Rightarrow yz = 6$$

Also  $(y - z)^2 = (y + z)^2 - 4yz$

$$\Rightarrow (y - z)^2 = (y + z)^2 - 4yz$$

$$\Rightarrow (y - z)^2 = 25 - 4(6) = 1$$

$$\Rightarrow y - z = 1$$
 ... (2)

from (1) and (2),  $y = 3$  and  $z = 2$

for calculating odd divisor of  $p = 2^x \cdot 3^y \cdot 5^z$

$x$  must be zero

$$P = 2^0 \cdot 3^3 \cdot 5^2$$

$\therefore$  total odd divisors must be  $(3 + 1)(2 + 1) = 12$

- 17.** Let  $f(x) = \sin^{-1} x$  and  $g(x) = \frac{x^2 - x - 2}{2x^2 - x - 6}$ . If  $g(2) = \lim_{x \rightarrow 2} g(x)$ , then the domain of the function  $f \circ g$  is :

- (1)  $(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$   
 (2)  $(-\infty, -1] \cup [2, \infty)$   
 (3)  $(-\infty, -2] \cup [-1, \infty)$   
 (4)  $(-\infty, -2] \cup \left[-\frac{3}{2}, \infty\right)$

**Ans. (1)**

Sol.  $g(2) = \lim_{x \rightarrow 2} \frac{(x-2)(x+1)}{(2x+3)(x-2)} = \frac{3}{7}$

For domain of  $f \circ g(x)$

$$\left| \frac{x^2 - x - 2}{2x^2 - x - 6} \right| \leq 1 \Rightarrow (x+1)^2 \leq (2x+3)^2 \Rightarrow 3x^2 + 10x + 8 \geq 0$$

$$\Rightarrow (3x+4)(x+2) \geq 0$$

$$x \in (-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$$

- 18.** If the mirror image of the point  $(1, 3, 5)$  with respect to the plane  $4x - 5y + 2z = 8$  is  $(\alpha, \beta, \gamma)$ , then  $5(\alpha + \beta + \gamma)$  equals:

- (1) 47  
 (2) 39  
 (3) 43  
 (4) 41

**Ans. (1)**

Sol. Image of  $(1, 3, 5)$  in the plane  $4x - 5y + 2z = 8$  is  $(\alpha, \beta, \gamma)$

$$\Rightarrow \frac{\alpha - 1}{4} = \frac{\beta - 3}{-5} = \frac{\gamma - 5}{2} = -2 \frac{(4(1) - 5(3) + 2(5) - 8)}{4^2 + 5^2 + 2^2} = \frac{2}{5}$$

$$\therefore \alpha = 1 + 4 \left(\frac{2}{5}\right) = \frac{13}{5}$$

$$\beta = 3 - 5 \left(\frac{2}{5}\right) = 1 = \frac{5}{5}$$

$$\gamma = 5 + 2 \left(\frac{2}{5}\right) = \frac{29}{5}$$

$$\text{Thus, } 5(\alpha + \beta + \gamma) = 5 \left(\frac{13}{5} + \frac{5}{5} + \frac{29}{5}\right) = 47$$

**19.** Let  $f(x) = \int_0^x e^t f(t) dt + e^x$  be a differentiable function for all  $x \in \mathbb{R}$ . Then

$f(x)$  equals.

(1)  $2e^{(e^x-1)} - 1$

(2)  $e^{(e^x-1)}$

(3)  $2e^{e^x} - 1$

(4)  $e^{e^x} - 1$

**Ans. (1)**

Sol. Given,  $f(x) = \int_0^x e^t f(t) dt + e^x \quad \dots(1)$

Differentiating both sides w.r.t  $x$

$$f'(x) = e^x \cdot f(x) + e^x \quad \text{(Using Newton Leibnitz Theorem)}$$

$$\Rightarrow \frac{f'(x)}{f(x) + 1} = e^x$$

Integrating w.r.t  $x$

$$\int \frac{f'(x)}{f(x) + 1} dx = \int e^x dx$$

$$\Rightarrow \ln(f(x) + 1) = e^x + c$$

Put  $x = 0$

$$\ln 2 = 1 + c \quad (\because f(0) = 1, \text{ from equation (1)})$$

$$\therefore \ln(f(x) + 1) = e^x + \ln 2 - 1$$

$$\Rightarrow f(x) + 1 = 2 \cdot e^{e^x-1}$$

$$\Rightarrow f(x) = 2e^{e^x-1} - 1$$

**20.** The triangle of maximum area that can be inscribed in a given circle of radius 'r' is:

(1) A right angle triangle having two of its sides of length  $2r$  and  $r$ .

(2) An equilateral triangle of height  $\frac{2r}{3}$ .

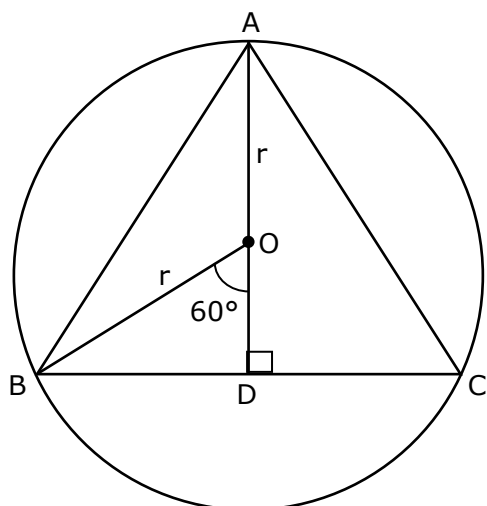
(3) An isosceles triangle with base equal to  $2r$ .

(4) An equilateral triangle having each of its side of length  $\sqrt{3} r$ .

**Ans. (4)**

Sol. Triangle of maximum area that can be inscribed in a circle is an equilateral triangle.

Let  $\Delta ABC$  be inscribed in circle,



Now, in  $\triangle OBD$

$$OD = r \cos 60^\circ = \frac{r}{2}$$

$$\text{Height} = AD = \frac{3r}{2}$$

Again in  $\triangle ABD$

$$\text{Now } \sin 60^\circ = \frac{3 \frac{r}{2}}{AB}$$

$$\Rightarrow AB = \sqrt{3} r$$

### Section - B

1. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is

**Ans. 1000**

Sol. Since, required number has G.C.D with 18 as 3. It must be odd multiple of '3' but not a multiple of '9'.

(i) Now, 4-digit number which are odd multiple of '3' are,  
1005, 1011, 1017, ..... 9999  $\rightarrow$  1499

(ii) 4-digit number which are odd multiple of 9 are,  
1017, 1035, ..... 9999  $\rightarrow$  499

$$\therefore \text{Required numbers} = 1499 - 499 = 1000$$

2. Let  $\alpha$  and  $\beta$  be two real numbers such that  $\alpha + \beta = 1$  and  $\alpha\beta = -1$ . Let  $P_n = (\alpha)^n + (\beta)^n$ ,  $P_{n-1} = 11$  and  $P_{n+1} = 29$  for some integer  $n \geq 1$ . Then, the value of  $P_n^2$  is \_\_\_\_\_.

**Ans. 324**

Sol. Given,  $\alpha + \beta = 1$ ,  $\alpha\beta = -1$

$\therefore$  Quadratic equation with roots  $\alpha, \beta$  is  $x^2 - x - 1 = 0$

$$\Rightarrow \alpha^2 = \alpha + 1$$

Multiplying both sides by  $\alpha^{n-1}$

$$\alpha^{n+1} = \alpha^n + \alpha^{n-1} \quad \text{_____ (1)}$$

Similarly,

$$\beta^{n+1} = \beta^n + \beta^{n-1} \quad \text{_____ (2)}$$

Adding (1) & (2)

$$\alpha^{n+1} + \beta^{n+1} = (\alpha^n + \beta^n) + (\alpha^{n-1} + \beta^{n-1})$$

$$\Rightarrow P_{n+1} = P_n + P_{n-1}$$

$$\Rightarrow 29 = P_n + 11 \quad (\text{Given, } P_{n+1} = 29, P_{n-1} = 11)$$

$$\Rightarrow P_n = 18$$

$$\therefore P_n^2 = 18^2 = 324$$

- 3.** Let  $X_1, X_2, \dots, X_{18}$  be eighteen observations such that  $\sum_{i=1}^{18} (X_i - \alpha) = 36$  and  $\sum_{i=1}^{18} (X_i - \beta)^2 = 90$ , where  $\alpha$  and  $\beta$  are distinct real numbers. If the standard deviation of these observations is 1, then the value of  $|\alpha - \beta|$  is \_\_\_\_\_.

**Ans. 4**

Sol. Given,  $\sum_{i=1}^{18} (X_i - \alpha) = 36$

$$\Rightarrow \sum x_i - 18\alpha = 36$$

$$\Rightarrow \sum x_i - 18(\alpha + 2) \quad \dots(1)$$

Also,  $\sum_{i=1}^{18} (X_i - \beta)^2 = 90$

$$\Rightarrow \sum x_i^2 + 18\beta^2 - 2\beta \sum x_i = 90$$

$$\Rightarrow \sum x_i^2 + 18\beta^2 + 2\beta \times 18(\alpha + 2) = 90 \quad (\text{using equation (1)})$$

$$\Rightarrow \sum x_i^2 = 90 - 18\beta^2 + 36\beta(\alpha + 2)$$

$$\sigma^2 = 1 \Rightarrow \frac{1}{18} \sum x_i^2 - \left( \frac{\sum x_i}{18} \right)^2 = 1 \quad (\because \sigma = 1, \text{ given})$$

$$\Rightarrow \frac{1}{18} (90 - 18\beta^2 + 36\alpha\beta + 72\beta) - \left( \frac{18(\alpha + 2)}{18} \right)^2 = 1$$

$$\Rightarrow 90 - 18\beta^2 + 36\alpha\beta + 72\beta - 18(\alpha + 2)^2 = 18$$

$$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - (\alpha + 2)^2 = 1$$

$$\Rightarrow 5 - \beta^2 + 2\alpha\beta + 4\beta - \alpha^2 - 4 - 4\alpha = 1$$

$$\Rightarrow \alpha^2 - \beta^2 + 2\alpha\beta + 4\beta - 4\alpha = 0$$

$$\Rightarrow (\alpha - \beta)(\alpha - \beta + 4) = 0$$

$$\Rightarrow \alpha - \beta = -4$$

$$\therefore |\alpha - \beta| = 4 \quad (\alpha \neq \beta)$$



4. In  $I_{m,n} = \int_0^1 x^{m-1} (1-x)^{n-1} dx$ , for  $m, n \geq 1$  and  $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \alpha I_{m,n}$ ,  $\alpha \in \mathbb{R}$ , then  $\alpha$  equals \_\_\_\_\_.

**Ans. 1**

Sol.  $I_{m,n} = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx$

Put  $x = \frac{1}{y+1} \Rightarrow dx = \frac{-1}{(y+1)^2} dy$

$1-x = \frac{y}{y+1}$

$\therefore I_{m,n} = \int_{\infty}^0 \frac{y^{n-1}}{(y+1)^{m+n}} (-1) dy = \int_0^{\infty} \frac{y^{n-1}}{(y+1)^{m+n}} dy \quad \dots(i)$

Similarly  $I_{m,n} = \int_0^1 x^{n-1} \cdot (1-x)^{m-1} dx$

$\Rightarrow I_{m,n} = \int_0^{\infty} \frac{y^{m-1}}{(y+1)^{m+n}} dy \quad \dots(ii)$

From (i) & (ii)

$2I_{m,n} = \int_0^{\infty} \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy$

$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy + \int_1^{\infty} \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy$   
 $\qquad \qquad \qquad I_1 \qquad \qquad \qquad I_2$

Put  $y = \frac{1}{z}$  in  $I_2$

$dy = -\frac{1}{z^2} dz$

$\Rightarrow 2I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy + \int_1^0 \frac{z^{m-1} + z^{n-1}}{(z+1)^{m+n}} (-dz)$

$\Rightarrow I_{m,n} = \int_0^1 \frac{y^{m-1} + y^{n-1}}{(y+1)^{m+n}} dy \Rightarrow \alpha = 1$

5. Let L be a common tangent line to the curves  $4x^2 + 9y^2 = 36$  and  $(2x)^2 + (2y)^2 = 31$ . Then the square of the slope of the line L is \_\_\_\_\_.

**Ans. 3**

Sol. E:  $\frac{x^2}{9} + \frac{y^2}{4} = 1$       C:  $x^2 + y^2 = \frac{31}{4}$

equation of tangent to ellipse is

$y = mx \pm \sqrt{9m^2 + 4} \quad \dots(i)$

equation of tangent to circle is

$$y = mx \pm \sqrt{\frac{31}{4}m^2 + \frac{31}{4}} \quad \dots(ii)$$

Comparing equation (i) & (ii)

$$9m^2 + 4 = \frac{31}{4}m^2 + \frac{31}{4}$$

$$\Rightarrow 36m^2 + 16 = 31m^2 + 31$$

$$\Rightarrow 5m^2 = 15$$

$$\Rightarrow m^2 = 3$$

6. If the matrix  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$  satisfies the equation

$$A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for some real numbers } \alpha \text{ and } \beta, \text{ then } \beta -$$

$\alpha$  is equal to \_\_\_\_\_.

**Ans. 4**

Sol.  $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⋮  
⋮  
⋮  
⋮  
⋮

$$A^{19} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{19} & 0 \\ 3 & 0 & -1 \end{bmatrix}, A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{L.H.S} = A^{20} + \alpha A^{19} + \beta A = \begin{bmatrix} 1 + \alpha + \beta & 0 & 0 \\ 0 & 2^{20} + \alpha 2^{19} + 2\beta & 0 \\ 3\alpha + 3\beta & 0 & 1 - \alpha - \beta \end{bmatrix}$$

$$\text{R.H.S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \alpha + \beta = 0 \text{ and } 2^{20} + \alpha 2^{19} + 2\beta = 4$$

$$\Rightarrow 2^{20} + \alpha(2^{19} - 2) = 4$$

$$\Rightarrow \alpha = \frac{4 - 2^{20}}{2^{19} - 2} = -2$$

$$\Rightarrow \beta = 2$$

$$\therefore \beta - \alpha = 4$$

7. If the arithmetic mean and geometric mean of the  $p^{\text{th}}$  and  $q^{\text{th}}$  terms of the sequence  $-16, 8, -4, 2, \dots$  satisfy the equation  $4x^2 - 9x + 5 = 0$ , then  $p+q$  is equal to \_\_\_\_\_.

**Ans. 10**

Sol. Given,  $4x^2 - 9x + 5 = 0$

$$\Rightarrow (x - 1)(4x - 5) = 0$$

$$\Rightarrow \text{A.M} = \frac{5}{4}, \text{G.M} = 1 \quad (\text{Q A.M} > \text{G.M})$$

Again, for the series

$$-16, 8, -4, 2, \dots$$

$$p^{\text{th}} \text{ term } t_p = -16 \left( \frac{-1}{2} \right)^{p-1}$$

$$q^{\text{th}} \text{ term } t_q = -16 \left( \frac{-1}{2} \right)^{q-1}$$

$$\text{Now, A.M} = \frac{t_p + t_q}{2} = \frac{5}{4} \text{ \& G.M} = \sqrt{t_p t_q} = 1$$

$$\Rightarrow 16^2 \left( -\frac{1}{2} \right)^{p+q-2} = 1$$

$$\Rightarrow (-2)^8 = (-2)^{(p+q-2)}$$

$$\Rightarrow p + q = 10$$

8. Let the normals at all the points on a given curve pass through a fixed point  $(a, b)$ . If the curve passes through  $(3, -3)$  and  $(4, -2\sqrt{2})$ , and given that  $a - 2\sqrt{2}b = 3$ , then  $(a^2 + b^2 + ab)$  is equal to \_\_\_\_\_.

**Ans. 9**

Sol. Let the equation of normal is  $Y - y = -\frac{1}{m}(X - x)$ , where,  $m = \frac{dy}{dx}$

As it passes through (a, b)

$$b - y = -\frac{1}{m}(a - x) = -\frac{dx}{dy}(a - x)$$

$$\Rightarrow (b - y)dy = (x - a)dx$$

$$by - \frac{y^2}{2} = \frac{x^2}{2} - ax + c \quad \dots(i)$$

It passes through (3, -3) & (4,  $-2\sqrt{2}$ )

$$\therefore -3b - \frac{9}{2} = \frac{9}{2} - 3a + c$$

$$\Rightarrow -6b - 9 = 9 - 6a + 2c$$

$$\Rightarrow 6a - 6b - 2c = 18$$

$$\Rightarrow 3a - 3b - c = 9 \quad \dots(ii)$$

Also

$$-2\sqrt{2}b - 4 = 8 - 4a + c$$

$$4a - 2\sqrt{2}b - c = 12 \quad \dots(iii)$$

$$\text{Also } a - 2\sqrt{2}b = 3 \quad \dots(iv) \text{ (given)}$$

$$(ii) - (iii) \Rightarrow -a + (2\sqrt{2} - 3)b = -3 \quad \dots(v)$$

$$(iv) + (v) \Rightarrow b = 0, \quad a = 3$$

$$\therefore a^2 + b^2 + ab = 9$$

**9.** Let z be those complex number which satisfy

$$|z+5| \leq 4 \text{ and } z(1+i) + \bar{z}(1-i) \geq -10, i = \sqrt{-1}.$$

If the maximum value of  $|z+1|^2$  is  $\alpha + \beta\sqrt{2}$ , then the value of  $(\alpha + \beta)$

is \_\_\_\_\_.

**Ans. 48**

**Sol.** Given,  $|z + 5| \leq 4$

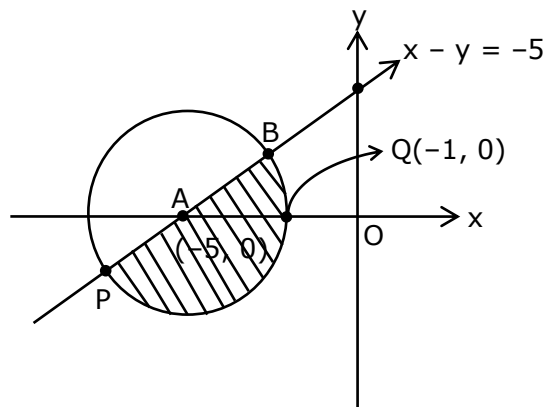
$$\Rightarrow (x + 5)^2 + y^2 \leq 16 \quad \dots(1)$$

$$\text{Also, } z(1+i) + \bar{z}(1-i) \geq -10.$$

$$\Rightarrow x - y \geq -5 \quad \dots(2)$$

From (1) and (2)

Locus of z is the shaded region in the diagram.



$|z + 1|$  represents distance of 'z' from  $Q(-1, 0)$

Clearly 'p' is the required position of 'z' when  $|z + 1|$  is maximum.

$$\therefore P \equiv (-5 - 2\sqrt{2}, -2\sqrt{2})$$

$$\therefore (PQ)^2 \Big|_{\max} = 32 + 16\sqrt{2}$$

$$\Rightarrow \alpha = 32$$

$$\Rightarrow \beta = 16$$

$$\text{Thus, } \alpha + \beta = 48$$

- 10.** Let  $a$  be an integer such that all the real roots of the polynomial  $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$  lie in the interval  $(a, a + 1)$ . Then,  $|a|$  is equal to \_\_\_\_\_.

**Ans. 2**

**Sol.** Let,  $f(x) = 2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$

$$\Rightarrow f'(x) = 10(x^4 + 2x^3 + 3x^2 + 2x + 1)$$

$$= 10 \left( x^2 + \frac{1}{x^2} + 2 \left( x + \frac{1}{x} \right) + 3 \right)$$

$$= 10 \left( \left( x + \frac{1}{x} \right)^2 + 2 \left( x + \frac{1}{x} \right) + 1 \right)$$

$$= 10 \left( \left( x + \frac{1}{x} \right) + 1 \right)^2 > 0; \forall x \in \mathbb{R}$$

$\therefore f(x)$  is strictly increasing function. Since it is an odd degree polynomial it will have exactly one real root.

Now, by observation

$$f(-1) = 3 > 0$$

$$f(-2) = -64 + 80 - 80 + 40 - 20 + 10$$

$$= -34 < 0$$

$$\Rightarrow f(x) \text{ has at least one root in } (-2, -1) \equiv (a, a + 1)$$

$$\Rightarrow a = -2$$

$$\Rightarrow |a| = 2$$